Recall: How to solve a first order linear ODE

① Find the standard form

$$y' + p(t)y = g(t)$$

② Find the integrating factor

$$\mu(t) = e^{\int p(t)\,dt}$$

③ Find the general solution

$$y(t) = \frac{\int \mu(t)\, g(t)\, dt + C}{\mu(t)}$$

In general, for nonlinear ODE, we don't know how to solve. Only some special types can be solved.

Separable ODE: $\frac{dy}{dx} = f(y)\, g(x)$

It can be solved by first separating the variables then integrating both sides.

$$\frac{dy}{dx} = f(y)\, g(x) \Rightarrow \frac{dy}{f(y)} = g(x)\, dx$$

$$\Rightarrow \int \frac{dy}{f(y)} = \int g(x)\, dx.$$

Example 1: $\dfrac{dy}{dx} = \dfrac{e^x - x}{e^{-y} + y}$

$$(e^{-y} + y)\,dy = (e^x - x)\,dx$$

$$\int (e^{-y} + y)\,dy = -e^{-y} + \frac{1}{2}y^2 = \int (e^x - x)\,dx = e^x - \frac{1}{2}x^2 + C$$

$$-e^{-y} + \frac{1}{2}y^2 = e^x - \frac{1}{2}x^2 + C$$ Implicit solution.

Example 2: $\dfrac{dy}{dx} = \dfrac{x^2 + \sin x}{y}, \quad y(0) = 1$

$$y\,dy = (x^2 + \sin x)\,dx.$$

$$\frac{1}{2}y^2 = \frac{1}{3}x^3 - \cos x + C.$$ (Implicit sol'n)

$$y(0) = 1 \Rightarrow \frac{1}{2}\cdot 1 = \frac{1}{3}\cdot 0 - 1 + C \Rightarrow C = \frac{3}{2}$$

$$y^2 = \frac{2}{3}x^3 - 2\cos x + 3$$

$$y = \pm\sqrt{\frac{2}{3}x^3 - 2\cos x + 3}$$ $\sqrt{x} > 0$

b/c $y(0) = 1 > 0$, y cannot be negative

$$y = \sqrt{\frac{2}{3}x^3 - 2\cos x + 3}$$ Explicit solution.

General Principle of Implicit sol'n & Explicit sol'n

* If you're asked to find general solution, then implicit solution is sufficient.

* If you're asked to solve an IVP, you should try to get explicit solution whenever possible.

Example 3: $y' = x y^3 (1+x^2)^{-1/2}$

$$\frac{dy}{dx} = y^3 \frac{x}{\sqrt{1+x^2}} \Rightarrow \frac{dy}{y^3} = \frac{x\,dx}{\sqrt{1+x^2}}$$

$$\int \frac{dy}{y^3} = \int y^{-3}\,dy = \frac{1}{-3+1} y^{-3+1}$$

$$\int x^p dx = \frac{1}{p+1} x^{p+1} + C$$

$$= -\frac{1}{2} y^{-2} = -\frac{1}{2y^2}$$

$$\int \frac{x\,dx}{\sqrt{1+x^2}} \cdot \frac{u=1+x^2}{du=2x\,dx} \int \frac{\frac{1}{2}\,du}{\sqrt{u}} = \frac{1}{2}\int u^{-\frac{1}{2}}\,du$$

$$= \frac{1}{2} \cdot \frac{1}{-\frac{1}{2}+1} u^{-\frac{1}{2}+1} + C$$

$$= u^{\frac{1}{2}} + C = \sqrt{1+x^2} + C$$

$$-\frac{1}{2y^2} = \sqrt{1+x^2} + C$$

Example 3': $y' = xy^3(1+x^2)^{-1/2}$ $\quad y(\sqrt{3}) = \frac{1}{2}$ $\quad (-\frac{1}{4})$

$$-\frac{1}{2y^2} = \sqrt{1+x^2} + C$$

$$-\frac{1}{2\times\frac{1}{2^2}} = \sqrt{1+3} + C \Rightarrow \underset{(-8)}{-2} = 2 + C \Rightarrow \underset{(-10)}{C = -4}$$

$$-\frac{1}{2y^2} = \sqrt{1+x^2} \underset{(-10)}{-4}$$

$$-\frac{1}{2(\sqrt{1+x^2}-4)} = y^2$$

$$y = \pm\sqrt{\frac{1}{\underset{(20)}{8} - 2\sqrt{1+x^2}}}$$

$y(\sqrt{3}) = \frac{1}{2} > 0$. can't be neg.

$\rightarrow (-\frac{1}{4} < 0)$

$$y = \underset{(-)}{}\sqrt{\frac{1}{\underset{(20)}{8} - 2\sqrt{1+x^2}}}$$

Remark 1: $\pm$ branch is not seen in the implicit solution

Remark 2: The interval of existence. (meaning the interval where the solution makes sense) is not seen in the implicit solution.

For the example above, $y \overset{(-)}{=} \sqrt{\frac{1}{\underset{(20)}{8} - 2\sqrt{1+x^2}}}$ make sense when

$$\underset{(20)}{8} - 2\sqrt{1+x^2} > 0$$

$$\Rightarrow \underset{(10)}{4} > \sqrt{1+x^2} \Rightarrow \underset{(100)}{16} > 1+x^2 \Rightarrow x^2 < \underset{(99)}{15} .$$

$\sqrt{f(x)}$ makes sense when $f(x) \geq 0$

$\Rightarrow -\sqrt{15} < x < \sqrt{15}$ ($-\sqrt{99}$, $\sqrt{99}$)

Review quadratic inequalities.

The interval of existence is $(-\sqrt{15}, \sqrt{15})$

Remark 3: For general solutions, the branch and the interval of existence makes no sense, b/c they depend on the choice of initial values. This is why implicit solution is enough.

Remark 4: Nevertheless, it's not always possible to find explicit solutions.

Existence & Uniqueness Theorem.

Motivating Example: $ty' + (t-1)y = -e^{-t}$, $y(0) = 1$

$y(0) = 0$

Std. form: $y' + \frac{t-1}{t} y = -\frac{e^{-t}}{t}$

Int. factor: $\mu(t) = e^{\int \frac{t-1}{t} dt}$

$$\int \frac{t-1}{t} dt = \int \left(1 - \frac{1}{t}\right) dt = t - \ln|t|$$

$$\mu(t) = e^{t - \ln|t|} \neq e^{t} - e^{\ln t}$$

abuse of algebra

$$= e^{t} \cdot e^{-\ln|t|}$$

$$= e^{t} \cdot \frac{1}{t} = \frac{e^{t}}{t}$$

$e^{a+b} = e^{a} \cdot e^{b}$. $e^{a-b} = e^{a+(-b)} = e^{a} \cdot e^{-b} = \frac{e^{a}}{e^{b}}$

Gen. sol'n: $y(t)=\dfrac{\int \frac{e^t}{t}\cdot\left(-\frac{e^{-t}}{t}\right)dt}{\frac{e^t}{t}}=\dfrac{t}{e^t}\cdot\int -\dfrac{1}{t^2}\,dt$

$$=\frac{t}{e^t}\cdot\left(\frac{1}{t}+c\right)=\frac{1}{e^t}+c\frac{t}{e^t}=e^{-t}+Cte^{-t}$$

$y(0)=1 \Rightarrow 1=e^0+C\cdot 0\cdot e^0 \Rightarrow 1=1$

This means arbitrary C satisfies the initial condition

In this case, the IVP has infinitely solutions.

$y(0)=0 \Rightarrow 0=e^0+C\cdot 0e^0 \Rightarrow 0=1$ impossible!

This means no C can satisfy the initial condition

In this case, the IVP has no sol'n.

Summary: An IVP might not have sol'ns. Also an IVP might have more than one sol'n. This happens when the IVP is pathologically posed.

Question: Is there any way to determine if an IVP is reasonably formulated.

Ans: YES.

Existence & Uniqueness Thm: linear version.

For the first order linear ODE in <u>std. form</u> w/ init. val.

$$y' + p(t)\,y = g(t), \quad y(t_0) = y_0$$

If ① Both $p(t)$ and $g(t)$ are continuous in an open interval (a,b)

② The open interval contains t_0, i.e. $a < t_0 < b$

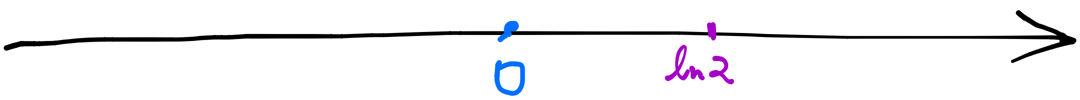
Then there exists a unique function $y = y(t)$ over the interval (a,b) that solves the IVP.

This theorem allows us to determine the interval of existence before solving the IVP.

Example 1: $ty' + (t-1)y = -e^{-t}, \quad y(\ln 2) = \frac{1}{2}$

Std. form $y' + \frac{t-1}{t}\,y = -\frac{e^{-t}}{t}$

blows up at $t = 0$. singular point



$\ln 2 > 0 \Rightarrow$ Interval of existence $(0, \infty)$

<u>Rmk</u>: The sol'n to the IVP is $y = e^{-t}$, exists everywhere. However, this doesn't change the fact that $t = 0$ is singular.

we don't want to worry about the existence near a singular pt.
The interval of existence obtained in this way is good enough.

General Steps:

1. Find the std. form.
2. Locate singular points (where $p(t)$ or $g(t)$ is not continuous)
3. Plot all singular points on the real line to get a bunch of intervals
4. Pick the interval where t_0 lies in.

Example 2: $(t-3)y' + (\ln t)y = 2t$. $y(1) = 2$

Std. form: $y' + \frac{\ln t}{t-3} y = \frac{2t}{t-3}$

blows up at $t=3$. t must be positive.

blows up $t=3$

$\frac{\ln t}{t-3}$ is defined when $t>0$ and $t \neq 3$. Continuous in the domain

$\frac{2t}{t-3}$ is defined and continuous when $t \neq 3$

Sing. pts: $t<0$, $t=3$

$0 < 1 < 3$, so interval of existeme is $(0,3)$

Example 3: $\sin 2t \, y' + \tan 4t \, y = \frac{1}{t}$, $y(\frac{\pi}{4}) = 0$

Std. form: $y' + \frac{\tan 4t}{\sin 2t} y = \frac{1}{t \sin 2t}$

$\frac{\tan 4t}{\sin 2t} = \frac{\sin 4t}{\cos 4t \cdot \sin 2t}$ is not continuous when $\cos 4t = 0$ or $\sin 2t = 0$

Recall: $\sin \alpha = 0$ when $\alpha = k\pi$. $k = 0, \pm 1, \pm 2, \cdots$

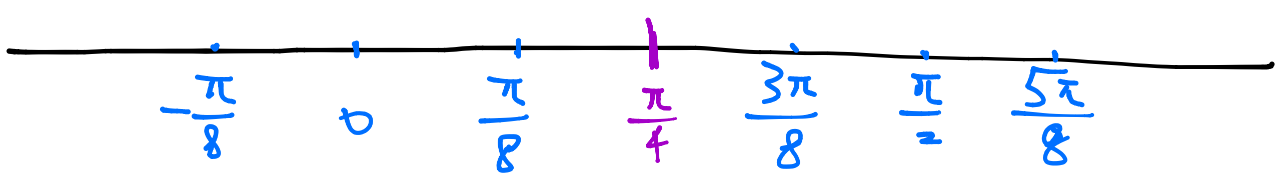
$\cos \alpha = 0$ when $\alpha = k\pi + \frac{\pi}{2}$ $k = 0, \pm 1, \pm 2$

$\sin 2t = 0 \Rightarrow 2t = k\pi \Rightarrow t = \frac{k\pi}{2}$, i.e., $t = 0, \pm\frac{\pi}{2}, \pm\pi, \pm\frac{3\pi}{2}, \cdots$

$\cos 4t = 0 \Rightarrow 4t = k\pi + \frac{\pi}{2} \Rightarrow t = \frac{\pi}{8} + \frac{k\pi}{4}$, i.e., $t = \pm\frac{\pi}{8}, \pm\frac{3\pi}{8}, \pm\frac{5\pi}{8}, \cdots$

$\frac{1}{t \sin 2t}$ is not continmous when $t = 0$ or $\sin 2t = 0$

the resulting singular points have been included above



Interval of existeme: $(\frac{\pi}{8}, \frac{3\pi}{8})$.

Attendance Quiz: ① Find sol'n to the IVP $y' = (1-2x)y^2$, $y(0) = -\frac{1}{6}$

Should find explicit sol'n & interval of existence.

② Find the interval of existence of the IVP

$$y' + \frac{t^4}{(t-2)^8} y = \sqrt{t}, \quad y(1) = 8.$$

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